26/3/17

Έστω $x_1, \ldots, x_n$ τδ από Πληθ. (μ, σ^2)

(i) Πληθυσμός κανονικός $\sum_1^n x_i \sim N(n\mu, n\sigma^2)$

$$\bar{x} \sim N(\mu, \sigma^2/n)$$

(ii) Πληθυσμός μη κανονικός : $\sum_1^n X_i \overset{\text{καν}}{\underset{\text{προσ}}{\sim}} N(n\mu, n\sigma^2)$

n μεγάλο ($n \geq 25$)

ΚΟΘ ή $\bar{X} \overset{\text{καν}}{\underset{\text{προσ.}}{\sim}} N(\mu, \sigma^2/n)$

Παράδειγμα 3.2

1 μsec, αριθμός σωμ. , $x \sim P(\lambda = 0.16)$

Σε 1 sec, πιθ να φτάσουν $[159000, 161000]$,

$1 sec = 10^6 \mu sec$

Έστω $x_1, x_2, \ldots$ σωμ. που θα φτάσουν στο

1°, 2°, ... μsec

$Y = X_1 + \ldots + X_{10^6}$ συνολικός αριθμός

$P(159000 \leq X \leq 161000)$

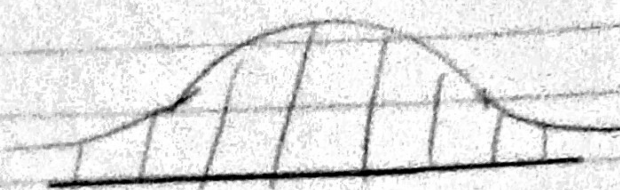
$X \sim P(\lambda = 0.16) \rightarrow E(x_i) = \lambda = 0,16 = Var(x_i)$

$Y \overset{\text{καν.}}{\underset{\text{ΚΟΘ}}{\sim}} N(n\cdot\mu = 10^6 * 0,16 ,\ n\cdot\sigma^2 = 10^6 * 0,16)$

$\simeq P(158999{,}5 \leq Y \leq 161000{,}5 \mid Y \overset{\text{καν}}{\underset{\text{απ}}{\sim}} N(160.000, 160.000))$

$= P\left(\frac{158999{,}5 - 160000}{\sqrt{160000}} \leq \frac{Y - \mu}{\sigma_Y} \leq \frac{161000{,}5 - 160.000}{\sqrt{160.000}} \,\middle|\, Z \overset{\text{καν.}}{\underset{\text{απ.}}{\sim}} N(0,1)\right)$

$= P(-2{,}5 \le Z \le 2{,}5) = 2 * P(0 \le Z \le 2{,}5) = 0{,}9876$



Κανονική Προσέγγιση της Διωνυμικής Κατανομής

n δοκιμές Bernoulli, με αποτέλεσμα της i-οστής δοκιμής

$X_i = 1$ για Ε, με $p = P(E)$

$= 0$ για Α, με $1 - p = q = P(A)$

$X = X_1 + X_2 + \dots X_n$

$X \sim B(n, p)$

$E(X) = np$

$Var(X) = npq$

$E(X_i) = p \cdot 1$, $E(X_i^2) = p$

$Var(X_i) = E(X_i^2) - (E(X_i))^2 = p - p^2 = p(1-p) = pq$ } Από ΚΟΘ $\longrightarrow$

$\rightarrow \quad X \overset{\text{καν}}{\underset{\text{κατ}}{\sim}} N(np, npq)$

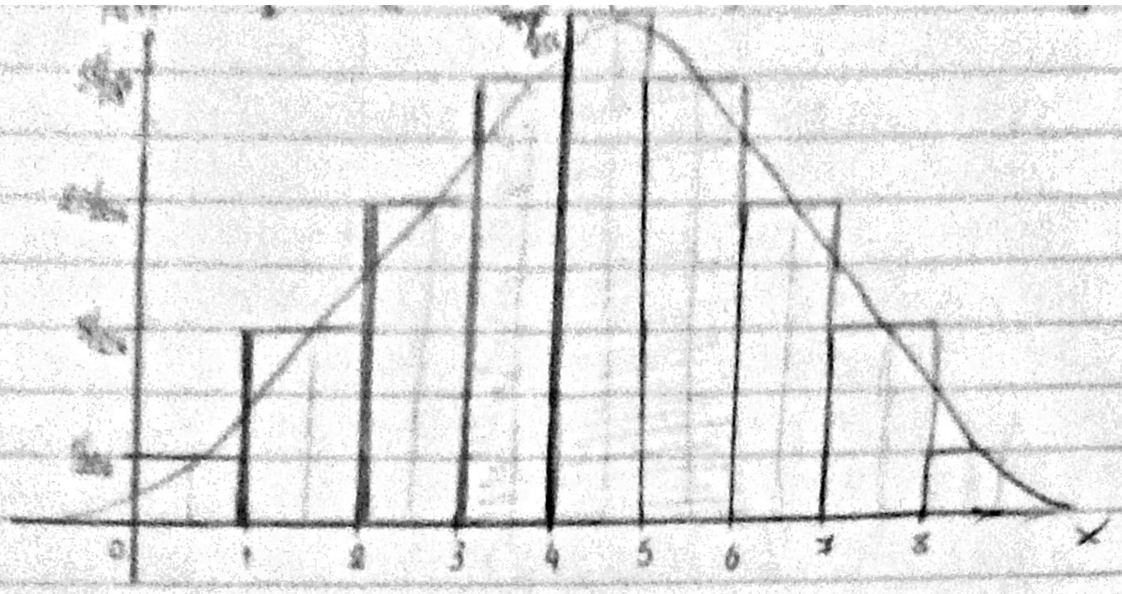
Άρα $\quad X \overset{\text{καν}}{\underset{\text{κατ}}{\sim}} N(np, npq)$

$X \sim B(n = 8, p = 0{,}5) \,/\, X \overset{\text{προσ}}{\sim} N(4, 2)$

$P(3 \le X \le 6) =$

$p_X(x) = \binom{8}{x} (0{,}5)^x (0{,}5)^{8-x}, \quad x = 0, 1, \dots 8$

$p_X(x) = \binom{8}{x} \left(\frac{1}{2}\right)^8$



~~P(3 ≤ x ≤ 6 | x ~ B(8~~

$P(3 \le x \le 6 \mid x \sim B(8;\ 0,5)) \approx$

$P(2,5 \le x \le 6,5 \mid x \overset{aprox}{\sim} N(4,2)) =$

$P\left(\frac{2,5-4}{\sqrt{2}} \le \frac{x-4}{\sigma_x} \le \frac{6,5-4}{\sqrt{2}} \mid z \sim N(0,1)\right)$

$P(-1,06 \le z \le 1,77) = 0,3554 + 0,4616 = 0,817$

Παράδειγμα 3.3

$p = P(A) = \frac{2}{2+2+1} = \frac{2}{5} = 0,4$

$n = 100$

$X(=\text{αρ. των } A) \sim B(n=100, p=0,4) \underset{\pi\rho\sigma}{\overset{\kappa\alpha}{=}} N(np=40, npq=24)$

$P(X \le 50) = \sum_{x=0}^{50} \binom{100}{x} (0,4)^x (0,6)^{100-x}$

$\simeq P(X \le 50,5 \mid X \overset{\kappa\alpha\tau}{\underset{\pi\rho\sigma}{\sim}} N(40,24)) =$

$= P\left(\frac{X-\mu_X}{\sigma_X}(=Z) \le \frac{50.4-40}{\sqrt{24}} \mid Z \underset{\pi\rho\sigma}{\overset{\kappa\alpha\tau}{\sim}} N(0,1)\right)$

$= P(Z \le 2,14) = 0,984,2$

Άσκηση 3.2

$X_1, \ldots, X_n$ τ.δ. $N(\mu, \sigma^2)$

(i) $X_i - \bar{X}$; (για i σταθερό)

(ii) $\frac{n(\bar{X}-\mu)}{\sigma^2} \sim X_1^2$

(iii) $E(S^2) = ;$

(iv) $\sum_1^n \frac{(X_i-\mu)^2}{\sigma^2} \sim X_n^2$

Λύση

(ii) $\frac{n(\bar{X}-\mu)^2}{\sigma^2} \sim X_1^2 \Leftarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \to \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \to$

$\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim N^2(0,1) = X_1^2$

(iii) $E(S^2) = ; \Leftarrow S^2 = \frac{1}{n}\sum_1^n (X_i-\bar{X})^2 = \frac{n-1}{n} S^2 \to$

$\to E(S^2) = \frac{n-1}{n} E(S^2) = \frac{n-1}{n} \cdot \sigma^2$

iii) $\sum_1^n \frac{(x_i-\mu)^2}{\sigma^2} \sim \chi_n^2$

$x_i \sim N(\mu, \sigma^2) \rightsquigarrow \frac{x_i-\mu}{\sigma} \sim N(0,1) \overset{\text{ανεξ}}{\rightsquigarrow} \sum_1^n \left(\frac{x_i-\mu}{\sigma}\right)^2 = \sum_1^n N^2(0,1) \sim$
$\sim \chi_n^2$

i) $Z = x_i - \bar{x} = \left(x_i - \frac{1}{n} x_i\right) - \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^n x_j$

$= \frac{n-1}{n} x_i - \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^n x_j$

$Z \sim N(E(Z), Var(Z))$

$E(Z) = \frac{n-1}{n}\mu - \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^n \mu = \frac{n-1}{n}\mu - \frac{n-1}{n}\mu = 0$

$Var(Z) = \left(\frac{n-1}{n}\right)^2 \sigma + \frac{1}{n^2} \sum_{\substack{j=1 \\ j \neq i}}^n \sigma^2 = \frac{(n-1)^2}{n^2}\sigma^2 + \frac{n-1}{n^2}\sigma^2$

$= \frac{n-1}{n^2}(n-1+1)\sigma^2 = \frac{n-1}{n}\sigma^2$

Κιτα λύσεις ...

$Var(x_i - \bar{x}) = Var x_i - Var \bar{x} = \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$

Άσκηση 3.4

x_1, x_2 τ.δ. από $N(0,1)$

(i) $\dfrac{x_2-x_1}{\sqrt{2}} \sim ;$

(ii) $\dfrac{(x_1+x_2)^2}{(x_2-x_1)^2} \sim ;$

(iii) $\dfrac{x_1+x_2}{\sqrt{(x_2-x_1)^2}} \sim ;$

(iv) $\dfrac{x_2^2}{x_1^2} \sim ;$

Λύση

(i) $x_2-x_1 \sim N(0-0,\ 1+1) \equiv N(0,2)$

$\dfrac{x_2-x_1-0}{\sqrt{2}} \sim N(0,1)$

(ii) $x_1+x_2 \sim N(0,2) \rightsquigarrow \dfrac{x_1+x_2}{\sqrt{2}} \sim N(0,1) \rightsquigarrow \dfrac{\left(\dfrac{x_1+x_2}{\sqrt{2}}\right)^2}{\left(\dfrac{x_2-x_1}{\sqrt{2}}\right)^2} \equiv$

$\equiv \dfrac{N^2(0,1)}{N^2(0,1)} \equiv \dfrac{\chi_1^2/1}{\chi_1^2/1} \equiv F_{1,1} \Big/ = \dfrac{(x_1+x_2)^2}{(x_2-x_1)^2}$

(iii) $\left.\begin{array}{l} \dfrac{x_1+x_2}{\sqrt{2}} \sim N(0,1) \\ \dfrac{x_2-x_1}{\sqrt{2}} \sim N(0,1) \end{array}\right\} \dfrac{\dfrac{x_1+x_2}{\sqrt{2}}}{\sqrt{\left(\dfrac{x_2-x_1}{\sqrt{2}}\right)^2}} = \left(\dfrac{N(0,1)}{\sqrt{\chi_1^2/1}} = t_1\right) = \dfrac{x_1+x_2}{\sqrt{(x_2-x_1)^2}}$

(iv) $\dfrac{x_2^2}{x_1^2} \sim ;$

iii) $\frac{X_2^2}{X_1^2} \sim ;$ $\left| \begin{array}{l} X_2 \sim N(0,1) \\ X_1 \sim N(0,1) \end{array} \right\} \to \frac{X_2^2/1}{X_1^2/1} \left(= \frac{X_1^2/1}{X_1^2/1} \right) \sim F_{1,1}$

Άσκηση 3:

$X_i \sim N(0,1)$ $i = 1, \ldots, 5$

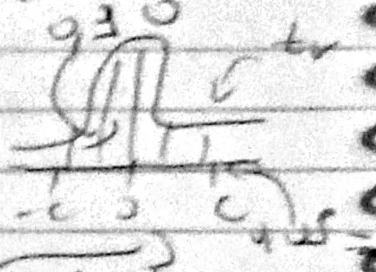
(i) $X_i^2 \sim \chi_1^2 \to Y = X_1^2 + \ldots + X_5^2 \sim \chi_5^2$

$E(Y) = E(\chi_5^2) = 5$, $Var(Y) = 2 \cdot 5 = 10$

(ii) $P\left(-c \leq \frac{2X_5}{\sqrt{X_1^2 + \ldots + X_5^2}} \leq c\right) = 0{,}90$

$\begin{array}{l} X_5 \sim N(0,1) \\ X_1^2 + \ldots + X_4^2 \sim \chi_4^2 \end{array} \Bigg\} T = \frac{X_5}{\sqrt{X_1^2 + \ldots + X_4^2 / 4}} = \frac{N(0,1)}{\sqrt{\chi_4^2/4}} \sim t_4$

$P(-c \leq T \leq c \mid T \sim t_4) = 0{,}90 \to$



$P(T \geq c) = 0{,}05 \to c = \ldots$

$c = t_{0,05,4} = 2{,}132$